Econ 201C: Problem Set 3

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1. **Auctions with Multiple Units**

   An auctioneer has k identical goods to sell. The seller has zero value for each good. There are n potential buyers, each with single-unit demand and IID private value $\theta \sim F[\underline{\theta}, \bar{\theta}]$. Assume the marginal revenue function

$$MR(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)}$$

   is increasing in θ.

   (a) Describe the direct revelation mechanism $\langle p_i, t_i \rangle$ in this setting. What are the constraints on these functions?

   Each buyer i reports a type $\widetilde{\theta}_i$, and let $\widetilde{\theta}$ be their concatenation. $p_i(\widetilde{\theta})$ is the probability buyer i gets allocated a good, and $t_i(\widetilde{\theta})$ is buyer i's transfer/payment to the seller. The allocation rule must satisfy feasibility

$$0 \leq p_i(\widetilde{\theta}) \leq 1 \quad \forall i, \widetilde{\theta}$$

   and the capacity constraint of the seller,

$$\sum_i p_i(\widetilde{\theta}) \leq k \quad \forall \widetilde{\theta}.$$

   Let

$$u_i(\theta_i, \widetilde{\theta}_i) = \mathbb{E}_{\theta_{-i}} \left[\theta_i p_i(\widetilde{\theta}_i, \theta_{-i}) - t_i(\widetilde{\theta}_i, \theta_{-i}) \right]$$

   be type θ_i's interim utility for reporting type $\widetilde{\theta}_i$. The mechanism should satisfy incentive-compatibility (i.e. truth-telling is optimal):

$$u_i(\theta_i, \theta_i) \geq u_i(\theta_i, \widetilde{\theta}_i) \quad \forall i, \theta_i, \widetilde{\theta}_i,$$

   and the mechanism must be individually rational, i.e.

$$u_i(\theta_i, \theta_i) \geq 0, \quad \forall i, \theta_i.$$

   (b) Derive the seller's revenue from any IC/IR mechanism. What mechanism maximizes the seller's revenue?

   Let $q_i(\theta_i) = \mathbb{E}_{\theta_{-i}}[p_i(\theta_i, \theta_{-i})]$ be the interim probability of i receiving a unit, and similarly, $T_i(\theta_i)$ be the interim expected transfer. Write $U_i(\theta_i) = \theta_i q_i(\theta_i) - T_i(\theta_i)$ as the interim utility. By IC and the envelope theorem, we have

$$U'(\theta_i) = q_i(\theta_i),$$

so by FTC,

$$U_i(\theta_u) = U_i(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} q_i(s)ds.$$

Writing the expected revenue from bidder i as welfare minus expected utility, we have that

$$\begin{aligned}\mathbb{E}[T_i(\theta_i)] &= \mathbb{E}\left[MR(\theta_i)q_i(\theta_i)\right] - U_i(\underline{\theta}) \\ &= \mathbb{E}\left[MR(\theta_i)p_i(\theta)\right]\end{aligned}$$

where $U_i(\underline{\theta})$ is optimally set to 0 to maximize revenue while satisfying IR for the lowest type. Summing across bidders and performing pointwise maximization w.r.t. θ, the seller chooses p_i to maximize

$$\sum_{i=1}^{n} MR(\theta_i)p_i(\theta)$$

subject to the feasibility and capacity conditions,

$$0 \leq p_i(\widetilde{\theta}) \leq 1 \quad \forall i, \widetilde{\theta},$$

$$\sum_i p_i(\widetilde{\theta}) \leq k \quad \forall \widetilde{\theta}.$$

Since MR is monotonic, the optimal allocations will be bang-bang, so long as $MR(\theta_i) \geq 0$. Let the reserve type $r = MR^{-1}(0)$, so the revenue-maximizing allocation is $p_i(\theta) = \mathbb{I}\left[\text{if } i \text{ is among the top } k \text{ types and } \theta_i \geq r\right]$.

Because this is a cutoff mechanism, the optimal transfer is a cutoff price. Let $\theta_{-i}^{(k)}$ be the k-th highest value among other bidders, then for i to get allocated a good, it must hold that $\theta_i \geq \max\{r, \theta_{-i}^{(k)}\}$. Thus, transfers are

$$t_i(\theta) = p_i(\theta)\max\{r, \theta_{-i}^{(k)}\},$$

which makes it so all types above the cutoff purchase.

(c) How can we implement the optimal mechanism using a sealed pay-your-bid auction? Derive agents' bidding function in such an auction.[1]

We can implement this mechanism with the rule that the highest k bidders above r win, and each winner pays their own bid.

For the bidding function, suppose all bidders use a strictly increasing symmetric bid function $b(\theta)$. Type θ wins if at most $k-1$ of the $n-1$ bidders have values above θ, so their probability of winning is

$$P(\theta) = \sum_{j=0}^{k-1} \binom{n-1}{j} [1-F(\theta)]^j F(\theta)^{n-1-j}.$$

[1]In a pay-your-bid auction the good is given to the highest k bidders who pay their bids. Losing bidders pay nothing. To get an explicit expression, you might use the fact that the probability of j successes out of m coin tosses is $\binom{m}{j}p^j(1-p)^{m-j}$, where p is the probability of success.

Thus, type θ's interim utility is

$$U(\theta) = (\theta - b(\theta)P(\theta).$$

By payoff equivalence (envelope + FTC) and the fact that type r gets 0 utility,

$$U(\theta) = (\theta - b(\theta))P(\theta) = \int_r^\theta P(s)ds,$$

and so the symmetric bidding strategy is

$$b(\theta) = \theta - \frac{\int_r^\theta P(s)ds}{P(s)}.$$

In this case of 1 good, this matches the first-price auction bidding strategy. Higher types shade their true valuation more because they are more likely to win.

(d) How can we implement the optimal mechanism in dominant strategies?

We can implement the optimal mechanism in dominant strategies using a second-price auction where bidding truthfully is weakly dominant for all bidders. Formally, the rule is to allocate the k goods to the highest k with bids at least r, where a winning bidder pays the threshold transfer as before $\max\{r, b_{-i}^{(k)}\}$ if they win. This makes it so if your value is above this cutoff, bidding truthfully would yield positive utility, and if your value is below the cutoff, winning the auction would result in negative utility, and bidding truthful prevents this. Thus, bidding truthfully, $b(\theta) = \theta$ is weakly dominant and implements the same optimal mechanism in (b).

2. **Public Goods Provision**

A firm is considering building a public good (e.g. a swimming pool). There are n agents in the economy, each with IID private value $\theta_i \in [0, 1]$. Agents' valuations have density $f(\theta_i)$ and distribution $F(\theta_i)$. Assume that

$$MR(\theta_i) = \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)}$$

is increasing in θ_i. The cost of the swimming pool is cn, where $c > 0$.

First suppose the government passes a law that says the firm cannot exclude people from entering the swimming pool. A mechanism thus consists of a build decision $p(\theta_1, \ldots, \theta_n) \in [0, 1]$ and a payment by each agent $t_i(\theta_1, \ldots, \theta_n) \in \mathbb{R}$. The mechanism must be individually rational and incentive compatible. [Note: When showing familiar results your derivation can be heuristic.]

(a) Consider an agent with type θ_i, whose utility is given by

$$\theta_i p - t_i$$

Derive her utility in a Bayesian incentive compatible mechanism $\langle p, t_i \rangle$.

We again define i's interim build probability $P_i(\theta_i) = \mathbb{E}_{\theta_{-i}}[p(\theta_i, \theta_{-i})]$ and interim expected payment $T_i(\theta_i) = \mathbb{E}_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})]$. Thus, i receives interim expected utility from truthful reporting of θ_i:

$$U_i(\theta_i) = \theta_i P_i(\theta_i) - T_i(\theta_i).$$

By IC and envelope theorem, we also have

$$U'(\theta_i) = P_i(\theta_i),$$

and using FTC,

$$U_i(\theta_i) = U_i(0) + \int_0^{\theta_i} P_i(s)ds = U_i(0) + \int_0^{\theta_i} \mathbb{E}_{\theta_{-i}}[p(s, \theta_{-i})]ds.$$

If we impose individual rationality, the mechanism must deliver at least 0 to the lowest type $\theta = 0$, so $U_i(0) = 0$ optimally.

(b) Given a build decision $p(\cdot)$, derive the firm's profits.

The firm's expected profit can be written as

$$\Pi = \mathbb{E}_\theta \left[\sum_i t_i(\theta) = cnp(\theta) \right]$$

where the firm pays cn if the pool is built. We can write the interim transfer from agent i as welfare minus utility, i.e.

$$T_i(\theta_i) = \theta_i P_i(\theta_i) - \int_0^{\theta_i} P_i(s)ds,$$

and performing the familiar taking of expectations and IBP, we have

$$\mathbb{E}[T_i(\theta_i)] = \mathbb{E}\left[MR(\theta_i)p(\theta)\right].$$

Plugging this into expected profit, we have

$$\Pi = \mathbb{E}\left[\left(\sum_{i=1}^n MR(\theta_i) - cn \right) p(\theta) \right].$$

(c) What is the firm's profit-maximizing build decision?

From the firm's profit maximization objective, we see the optimal build decision is bang-bang: build (set $p(\theta) = 1$) when the sum of agents' marginal revenues is greater than total cost cn, and 0 otherwise.

(d) Show that $E[MR(\theta_i)] = 0$.

Let $\underline{\theta} = 0$. We write

$$\begin{aligned} E[MR(\theta)] &= \int_{\underline{\theta}}^{\bar{\theta}} f(\theta) \left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) d\theta \\ &= \int_{\underline{\theta}}^{\bar{\theta}} f(\theta)\theta \, d\theta - \int_{\underline{\theta}}^{\bar{\theta}} 1 - F(\theta) \, d\theta. \end{aligned}$$

Using integration by parts, the second term is

$$\int_{\underline{\theta}}^{\bar{\theta}} 1 - F(\theta)\, d\theta = [\theta(1 - F(\theta)]_{\underline{\theta}}^{\bar{\theta}} + \int_{\underline{\theta}}^{\bar{\theta}} \theta f(\theta)\, d\theta.$$

Cancelling and then evaluating the boundary terms, we get that

$$E[MR(\theta)] = \bar{\theta}(1 - F(\bar{\theta}) - \underline{\theta}(1 - F(\underline{\theta})) = \underline{\theta}.$$

(e) Show that as $n \to \infty$, the probability of provision goes to zero.

We have that the pool is provided iff

$$\frac{1}{n}\sum_{i=1}^{n} MR(\theta_i) \geq c.$$

Since θ_i are IID, we have by the LLN, the empirical average marginal revenue approaches the true expectation, which is 0 as calculated in (d). So long as $c > 0$ we have

$$\Pr\left(\frac{1}{n}\sum_{i=1}^{n} MR(\theta_i) \geq c\right) \to_{n\to\infty} 0.$$

(f) Solve for the firm's optimal build decision $p(\cdot)$ and participation rule $x_i(\cdot)$.

Firm can choose x_i which dictates whether i can participate in the public good, so utility is

$$u_i(\theta) = \theta_i x_i(\theta) p(\theta) - t_i(\theta),$$

and the allocation probability for i is now $q_i(\theta) = x_i(\theta)p(\theta)$. By the same reasoning as (a) and (b), we derive the firm's expected profits

$$\Pi = \mathbb{E}\left[\left(\sum_{i=1}^{n} MR(\theta_i)x_i(\theta) - cn\right) p(\theta)\right].$$

From this objective, the firm would want to include i if their MR is positive, i.e. $MR(\theta) \geq 0$. By monotonicity, let $r = MR^{-1}(0)$, and so the firm sets

$$x_i(\theta) = \mathbb{I}[\theta_i \geq r]$$

. Then, the firm implements the same bang-bang build decision, where

$$p(\theta) = \mathbb{I}[\sum_i \max\{MR(\theta_i), 0\} \geq cn]$$

.

(g) Suppose $n \to \infty$. Show there exists a cutoff c^* such that the firm provides the pool with probability one if $c < c^*$, and with probability zero if $c > c^*$.

Intuitively, the expected value of MR is no longer 0 when MR is conditioned to be nonnegative. Formally, let

$$c^\star = \mathbb{E}\left[\max\{MR(\theta_i), 0\}\right].$$

Thus, by the LLN, we have

$$\frac{1}{n}\sum_{i=1}^{n} \max\{MR(\theta_i), 0\} \to_{n\to\infty} c^\star.$$

Thus, if $c < c^\star$, the pool will be provided with probability 1, and vice versa of $c > c^\star$.

3. **Auctions with Correlated Values**

A principal has a single good. Two agents $i \in \{1, 2\}$ have values $v_i \in \{1, 2\}$ for the good. These values are correlated with joint distribution $f(v_1, v_2)$ given by the table below:

$f(v_1, v_2)$	$v_2 = 1$	$v_2 = 2$
$v_1 = 1$	1/3	1/6
$v_1 = 2$	1/6	1/3

First, we suppose the seller runs a second-price auction without reserve (and flips a fair coin in the case of a tie).

(a) Show that it is a weakly dominant strategy for both types of agents to bid their true value, $B_i = v_i$, for $v_i \in \{1, 2\}$ and $i \in \{1, 2\}$.

Let i's value be v_i, and let the other agent's bid be B_j. We have the following cases:

1) $v_i > B_j$. Then bidding truthfully wins i the auction, and yields positive utility $v_i - B_j$.

2) $v_i < B_j$. Then bidding truthfully loses i the auction yielding utility 0, but this is optimal, since a winning outcome would yield negative utility $v_i - B_j$.

3) $v_i = B_j$. Here, bidding anything, win or lose, nets i a utility of 0, so truthful bidding is still optimal.

(b) Assume agents bid their true value. What is the interim expected utility of each type?

Assume i has type $v_i = 1$. Then j has $v_j = 1$ with probability 2/3 and $v_j = 2$ with probability 1/3. Thus, interim expected utility is

$$U_i(1) = \frac{2}{3}(0) + \frac{1}{3}(0) = 0.$$

If i has type $v_i = 2$, then j has 1 with probability 1/3 and 2 with probability 2/3. Interim expected utility is thus

$$U_i(2) = \frac{1}{3}(1) + \frac{2}{3}(0) = \frac{1}{3}.$$

Now, suppose we supplement the auction with a "side payment". If agent 2 bids $B_2 = 1$, then agent 1 is paid α by the principal; whereas if $B_2 = 2$ then agent 1 pays β to the principal.[2] Similarly, if agent 1 bids $B_1 = 1$ then agent 2 is paid α whereas if $B_1 = 2$ then agent 2 pays β.

(c) Show that it is a weakly dominant strategy for both types of agents to bid their true value, $B_i = v_i$, for $v_i \in \{1, 2\}$ and $i \in \{1, 2\}$.

Since side payments only depend on the other player's bid, when agent i chooses bid B_i, this side payment is fixed depending on B_j. Since an additional constant term only shifts all utilities, we still have that truthful bidding is weakly dominant by the logic of (a).

(d) Assume agents bid their true value. If the seller picks the side-payment judiciously she can fully extract from both types of agents.[3] What are the optimal (α, β)?

We impose binding IR conditions on the equations in (b). Assume i has type $v_i = 1$. Then j has $v_j = 1$ with probability $2/3$ and $v_j = 2$ with probability $1/3$. Thus, interim expected utility is

$$U_i(1) = \frac{2}{3}(\alpha) - \frac{1}{3}(\beta) = 0.$$

If i has type $v_i = 2$, then j has 1 with probability $1/3$ and 2 with probability $2/3$. Interim expected utility is thus

$$U_i(2) = \frac{1}{3}(1 + \alpha) - \frac{2}{3}(\beta) = 0.$$

Solving the system of equations gives $\alpha = 1/3$, $\beta = 2/3$.

4. **Adverse Selection in Labor Markets**

Firm A employs a worker with productivity θ, and observes the worker's productivity. Two firms, Firm B and C, would like to employ the worker. Unlike Firm A, Firms B and C do not observe the worker's productivity and think $\theta \sim F[0, 1]$. In the game, all three firms simultaneously choose wages. Denote Firm A's wage by w and Firm B and C's wages by x_B and x_C, and let $x = \max\{x_B, x_C\}$ be the highest outside offer (and call this the "market wage"). The worker quits Firm A and accepts the highest outside offer with probability

$$q = \begin{cases} 1 & w < x \\ \mu & w \geq x \end{cases}$$

In words, the worker always quits if Firm A's offer is worse than the market wage, and quits with probability $\mu < 1$ if Firm A's offer is better. When Firm A employs the worker (i.e., when the worker does not quit) they earn profit $\pi = \theta - w$, and similarly for Firms B and C. We are interested in the pure BNE of this wage-posting game. To this end:

[2] Intuitively, if agent 2's value is high then agent 1's value is likely to be high and we should charge them more. If agent 2 bids $B_2 \notin \{1, 2\}$, the side payment of agent 1 can be arbitrary (this is off-path).

[3] Formally, the mechanism is interim Bayesian individually rational, so both types earn zero expected interim utility given the joint distribution of types $f(v_1, v_2)$.

(a) Given market wage x and worker productivity θ, what is Firm A's optimal wage w? (Note, if Firm A does not wish to employ the worker they can always choose $w = 0$).

If the firm wants to keep the worker, they choose $w = x$ to minimize costs, which gives expected profit

$$(1 - \mu)(\theta - x).$$

The firm will implement this if this expected profit is nonnegative, otherwise, they post $w = 0$ and let the worker quit. This is equivalent to $w(\theta, x) = x \cdot \mathbb{I}[\theta \geq x]$.

(b) Given market wage x, provide a formula for the expected quality of the workers who quit Firm A, $E[\theta|\text{quit}]$?

From Firm A's optimal strategy, we know a worker quits Firm A with probability 1 if $\theta < x$, which they believe occurs with probability $F(x)$, or with probability μ if $\theta \geq x$, which they believe happens with probability $1 - F(x)$. Thus, we can write

$$\begin{aligned}
E[\theta|\text{quit}] &= E[\theta|\theta < x, \text{quit}] \cdot \Pr(\theta < x \mid \text{quit}) + E[\theta|\theta \geq x, \text{quit}] \cdot \Pr(\theta \geq x \mid \text{quit}) \\
&= \frac{1}{F(x)} \int_0^x \theta f(\theta)\, d\theta \cdot \frac{F(x)}{F(x) + \mu(1 - F(x)} + \frac{1}{1 - F(x)} \int_x^1 \theta f(\theta)\, d\theta \cdot \frac{\mu(1 - F(x))}{F(x) + \mu(1 - F(x)} \\
&= \int_0^x \theta f(\theta)\, d\theta \cdot \frac{1}{F(x) + \mu(1 - F(x)} + \int_x^1 \theta f(\theta)\, d\theta \cdot \frac{\mu}{F(x) + \mu(1 - F(x)}
\end{aligned}$$

(c) Thinking of Firms B and C as two Bertrand competitors, argue that the market wage satisfies

$$x = E[\theta|\text{quit}].$$

Suppose that $x < E[\theta|\text{quit}]$. Then the firm who ends up hiring the quitting worker has ex-ante profit

$$E[\theta|\text{quit}] - x > 0.$$

Since this is profitable, the firm that has the smaller wage (and netted 0 profit) will want to bid a wage $x' = x + \varepsilon$, which will then capture positive profit.

If on the other hand, $x > E[\theta|\text{quit}]$, then the firm who hires the quitter receives a negative ex-ante profit, so they would want to lower the wage or not hire at x.

(d) Suppose $\theta \sim U[0, 1]$. What is the market wage x?

We have

$$\begin{aligned}
x &= E[\theta|\text{quit}] \\
&= \int_0^x \theta\, d\theta \cdot \frac{1}{x + \mu(1 - x)} + \int_x^1 \theta\, d\theta \cdot \frac{\mu}{x + \mu(1 - x)} \\
&= \frac{(1/2)x^2 + (1/2)\mu(1 - x^2)}{x + \mu(1 - x)} \\
\implies (1 - \mu)x^2 + \mu x &= (1/2)\mu + (1/2)(1 - \mu)x^2 \\
\implies (1 - \mu)x^2 + 2\mu x - \mu &= 0.
\end{aligned}$$

Solving the quadratic and taking the feasible solution gives

$$x = \frac{\sqrt{\mu}}{1 + \sqrt{\mu}}.$$

(e) Suppose $\theta \sim U[0,1]$. How does the market wage x vary as μ increases from 0 to 1? Provide an intuition.

The market wage at $\mu = 0$ is $x = 0$, and rises to $x = 1/2$ when $\mu = 1$. $x(\mu)$ is increasing as well:

$$x'(\mu) = \frac{1}{2\sqrt{\mu}(1 + \sqrt{\mu})^2} > 0.$$

This models adverse selection. When μ is low, Firm A retains high productivity θ workers well by setting $w = x$, so a larger proportion of quitters are low-productivity workers. Firms B and C realize this and set a lower market wage. In the extreme case where Firm A retains all high-productivity workers with $w = x$ ($\mu = 0$), Firms B and C know they are only receiving low-productivity quitters, and the market wage collapses down to 0.

Vice versa, when μ is large, Firm A cannot retain high-productivity workers well, even when matching $w = x$, so Firms B and C know a larger share of the quitters are high-productivity. In the extreme case where the worker quits no matter what (μ_1), the market wage is simply the expectation of θ which is $1/2$.